MTH 304: Metric Spaces and Topology

Practice Assignment III: Compactness

1. Reading assignment: Read the proofs of following assertions from section 1.12 - (ix), (xix), (xvii), (xxiv), (xxvii), & (xxix).

2. Show that a compact subspace of a metric space is bounded. Given a counterexample to show that the converse does not hold true.

3. Let X be a space, and Y be a compact space.

   (a) Show that the projection map $\pi_1 : X \times Y \to X$ is a closed map.

   (b) Let Y be Hausdorff space and $f : X \to Y$ be a map. Then f is continuous if, and only if, the *graph of f*

   $$G_f = \{(x, y) \,|\, (x, y) \in X \times Y \text{ and } y = f(x)\}$$

   is a closed subset of $X \times Y$.

4. Let $p : X \to Y$ be a continuous surjective map such that each fiber of f is compact (such a map is called a *perfect map*). Show that if Y is compact, then X is compact.

5. Show that for $n \geq 1$, S^n is the one point compactification of $\mathbb{R}^n$.

6. In each of the following topologies on $\mathbb{R}$, determine whether the closed interval $[a, b]$ is compact, limit point compact, sequentially coompact, and locally compact.

   (a) Cofinite topology.

   (b) Cocomplement topology.

   (c) $\mathbb{R}_\ell$.

   (d) $\mathbb{R}_K$.

7. A space X is a said to *countably compact* if every countable open covering of X has a finite subcovering. Show that in a T_1 space, countable compactness is equivalent to limit point compactness.

8. Let (X, d_x) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is called an *isometry* if

   $$d_X(a, b) = d_Y(f(a), f(b)), \ \forall a, b \in X.$$

   If X be compact, and $f : X \to Y$ is a surjective isometry, then show that f is a homeomorphism.

9. Let G be a topological group.

   (a) If $A \subset G$ is closed and $B \subset G$ is compact, then show that AB is closed.

   (b) If $H \leq G$ and H is compact, then show that the quotient map $G \to G/H$ is a closed map.

   (c) If $H \leq G$, H is compact, and G/H is compact, then show that G is compact.

10. Let (X, d) be a metric space. A map $f : X \to X$ is called a *contraction* if there exists $0 < \alpha < 1$ such that

$$d(x, y) \leq \alpha d(f(x), f(y)), \ \forall x, y \in X.$$

Show that a contraction $f : X \to X$ has a fixed point.

11. Let X and Y be locally compact Hausdorff spaces, and let $f : X \to Y$ be a homeomorphism.

    (a) Show that f extends to a homeomorphism of the one point compactifications of X and Y.

    (b) Show that the one point compactification of $\mathbb{N}$ is homeomorphic to $\{1/n\}_{n\in\mathbb{N}} \cup \{0\}$.

12. Prove or disprove the following statements.

    (a) Continuous image of a limit point compact space is limit point compact.

    (b) Continuous image of a locally compact space is locally compact.

    (c) Continuous image of a sequentially compact space is sequentially compact.